ROTATION OF A RIGID BODY WITH ROTATING FLYWHEELS ABOUT ITS CENTER OF INERTIA

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The problem of rotation of a rigid body about its center of inertia in presence of steady cyclic rotations within that body without however influencing its mass distribution, was first stated and investigated by Volterra in a number of papers over the period 1895 – 1899, and the basic results are given in his book "On the Theory of Latitude Variation" [1]. He proves the fundamental integrability of equations for the direction cosines of the body, in terms of elliptic functions of time. A particular case of Volterra's problem was investigated in more detail in the paper by Ena [2], in which he examines the motion of an asymmetric rigid body about one of the principal axes of inertia; a steady internal rotation of the body takes place, analogous to the uniform rotation of a flywheel.

In the present paper we investigate another particular case of Volterra's problem, when the body in question has a dynamic axial symmetry, and contains inside it a system of flywheels, rotating with velocities constant with respect to the body. The problem of motion of a body when the flywheels rotate with velocities constant with respect to inertial space, can also be reduced to the present case. We also assume that the total kinetic momentum of the system is constant during the motion, i.e. external perturbations are absent. Following the analysis of equations of the problem, a geometrical representation of the motion is given, possible types of motion of the body are shown and their dependence on the parameters of the system and on the initial conditions, is established. It is also shown, that the trajectory of the axis of symmetry describes, on the surface of the unit sphere, looped curves analogous to those met in Lagrange's problem.

1. Initial relationships. Let a system of n flywheels be situated within the frame of the carrying body, and let their axes be fixed in the frame and defined in terms of direction cosines $\alpha_k \beta_k \gamma_k$ (where k denotes the kth flywheel) relative to the principal xyz-axes of the frame. Then, assuming that the moments of inertia of the body (together with the flywheels) relative to its principal axes x, y and z are A, B, and C, respectively, and that the projections of the corresponding angular momenta of the flywheels on those axes are H_x, H_y and H_z, we can write the equations of conservation of total angular momentum as follows:

$$A\omega_x + H_x = L\sin\varphi\sin\vartheta, \qquad B\omega_y + H_y = L\cos\varphi\sin\vartheta$$

$$C\omega_z + H_z = L\cos\vartheta \tag{1.1}$$

$$H_x = \sum_{k=1}^{n} H_k\alpha_k, \qquad H_y = \sum_{k=1}^{n} H_k\beta_k, \qquad H_z = \sum_{k=1}^{n} H_k\gamma_k \qquad (H_k = I_k\Omega_k)$$

Here I_k is the moment of inertia of the kth flywheel, Ω_k is its angular velocity relative to the body, $\omega_x \omega_y \omega_z$ are the projections of the angular velocity vector on its principal axes, while φ, ϑ and ψ are Eulerian angles, defining the position of the x, y and z-axes relative to fixed $\xi\eta\zeta$-axes, in which the ζ-axis coincides with the vector of total kinetic momentum $\mathbf{L}$ of the system. Replacing in (1.1) the angular velocities with the corresponding expressions in terms of Euler angles and their derivatives and solving it with respect to these derivatives, we arrive at

$$\varphi^{\cdot} = \left[\frac{1}{C} - \left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right)\right] L\cos\vartheta + \cot\vartheta\left(\frac{H_x}{A}\sin\varphi + \frac{H_y}{B}\cos\varphi\right) - \frac{H_z}{C}$$

$$\vartheta^{\cdot} = \left(\frac{1}{A} - \frac{1}{B}\right) L\sin\vartheta\sin\varphi\cos\varphi - \left(\frac{H_x}{A}\cos\varphi - \frac{H_y}{B}\sin\varphi\right) \tag{1.2}$$

$$\psi^{\cdot} = \left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) L - \frac{1}{\sin\vartheta}\left(\frac{H_x}{A}\sin\varphi + \frac{H_y}{B}\cos\varphi\right)$$

This system can be further integrated in two particular cases of rotation of the flywheels. First of them, considered initially by Volterra [1], corresponds to the case when the velocities of rotation of the flywheels are maintained constant with respect to the body, i.e. $H_k =$ const , and consequently, H_x, H_y, $H_z =$ const . The other case for which exact integration of (1.2) is possible, is characterized by the fact, that the absolute velocities of rotation of the flywheels about their axes, are constant.

To find the necessary integrals for each of these cases, we shall write the system of Volterra's dynamic equations for our system [3]

$$A\omega_x^{\cdot} + (C - B)\,\omega_y\omega_z + \sum_{k=1}^{n} I_k\,[\Omega_k^{\cdot}\alpha_k + \Omega_k(\omega_y\gamma_k - \omega_z\beta_k)] = 0$$

$$B\omega_y^{\cdot} + (A - C)\,\omega_z\omega_x + \sum_{k=1}^{n} I_k\,[\Omega_k^{\cdot}\beta_k + \Omega_k(\omega_z\alpha_k - \omega_x\gamma_k)] = 0 \tag{1.3}$$

$$C\omega_z^{\cdot} + (B - A)\,\omega_x\omega_y + \sum_{k=1}^{n} I_k\,[\Omega_k^{\cdot}\gamma_k + \Omega_k(\omega_x\beta_k - \omega_y\alpha_k)] = 0$$

and the equations of rotation of the flywheels

$$I_k(\Omega_k^{\cdot} + \omega_x^{\cdot}\alpha_k + \omega_y^{\cdot}\beta_k + \omega_z^{\cdot}\gamma_k) = m_k \tag{1.4}$$

Here m_k is the angular momentum at the shaft of the kth flywheel. Assuming now that $\Omega_k =$ const , multiplying the equations of (1.3) by ω_x, ω_y and ω_z , respectively, and adding them, we obtain the following integral

$$\tfrac{1}{2} A\omega_x^2 + \tfrac{1}{2} B\omega_y^2 + \tfrac{1}{2} C\omega_z^2 = \text{const} \tag{1.5}$$

In the second case it is enough to put $m_k = 0$ in (1.4), after which, performing operations analogous to the above, it is easy to obtain, from (1.3), the integral

$$\tfrac{1}{2} A\omega_x^2 + \tfrac{1}{2} B\omega_y^2 + \tfrac{1}{2} C\omega_z^2 - \tfrac{1}{2}\sum_{k=1}^{n} I_k (\omega_x\alpha_k + \omega_y\beta_k + \omega_z\gamma_k)^2 = \text{const} \tag{1.6}$$

It can easily be shown that (1.6) is an energy integral. Indeed, if we write the expression for kinetic energy of the system in the form [3]

$$T = \tfrac{1}{2} A\omega_x^2 + \tfrac{1}{2} B\omega_y^2 + \tfrac{1}{2} C\omega_z^2 + \\ + \tfrac{1}{2}\sum_{k=1}^{n} I_k [\Omega_k^2 + 2\Omega_k (\omega_x\alpha_k + \omega_y\beta_k + \omega_z\gamma_k)] \tag{1.7}$$

and take into account the fact that when $m_k = 0$ then from (1.4) follows

$$I_k (\Omega_k + \omega_x\alpha_k + \omega_y\beta_k + \omega_z\gamma_k) = h_k = \text{const} \tag{1.8}$$

then we shall obtain the following expression for T :

$$2T = A\omega_x^2 + B\omega_y^2 + C\omega_z^2 - \sum_{k=1}^{n} I_k (\omega_x\alpha_k + \omega_y\beta_k + \omega_z\gamma_k)^2 + \sum_{k=1}^{n} \frac{h_k^2}{I_k} \tag{1.9}$$

which differs from the integral (1.6) by a constant term.

Using (1.8) we can represent the vector $\mathbf{H}(H_x H_y H_z)$ for the given type of rotation of the flywheels, in the form

$$\mathbf{H} = \sum_{k=1}^{n} \mathbf{h}_k - \mathbf{G}\cdot\omega \tag{1.10}$$

where tensor $\mathbf{G}$ is specified by the matrix of its components along the principal axes

$$\mathbf{G} = \begin{Vmatrix} \sum I_k\alpha_k^2 & \sum I_k\alpha_k\beta_k & \sum I_k\alpha_k\gamma_k \\ \sum I_k\beta_k\alpha_k & \sum I_k\beta_k^2 & \sum I_k\beta_k\gamma_k \\ \sum I_k\gamma_k\alpha_k & \sum I_k\gamma_k\beta_k & \sum I_k\gamma_k^2 \end{Vmatrix} \tag{1.11}$$

where the summation under the Σ sign is performed from 1 to n . Then, the law of conservation of angular momentum of the system can be written as

$$(\Theta - \mathbf{G})\cdot\omega + \sum_{k=1}^{n} \mathbf{h}_k = \mathbf{L} \tag{1.12}$$

where Θ is the inertia tensor of the body with the flywheels, defined on the principal axes by its components A, B and C .

Each of Equations (1.12) will contain in its projections on the principal axes, all three components of the vector ω . Consequently, in order to obtain from (1.12) three scalar equations each containing one component of ω , we must make use of another $x'y'z'$ coordinate system, rotated with respect to the principal axes of the body, The matrix of rotation of xyz-axes to $x'y'z'$-axes can be obtained in a manner identical to that, used to determine the principal axes of inertia of the body. We should note, that in the system of xyz-axes which may be called quasi-principal axes of the body, the form of (1.12) will be analogous to that of (1.1). Physical meaning however of the parameters A, B and C together with H_x, H_y and H_z

will be altered somewhat. The coordinate transformation $xyz \to x'y'z'$ results in the expression for kinetic energy assuming its normal form, in which the products of various components of the vector of angular velocity ω of the body, do not appear.

Both of the above cases can be described by the system of equations of the form (1.2) and possess the integral of the form (1.5). Assumimg the Eulerian angles to be the basic variables defining the position of principal (or quasi-principal) axes of the body relative to the $\xi\eta\zeta$-axes and replacing the constants H_x, H_y and H_z with R, μ and ν according to the following relationships,

$$H_x = AR \sin\mu \sin\nu, \qquad H_y = BR\cos\mu\sin\nu, \qquad H_z = CR\cos\nu$$
$$R = [(H_x/A)^2 + (H_y/B)^2 + (H_z/C)^2]^{1/2}$$

we can write (1.2) as

$$\begin{aligned}\varphi^{\cdot} &= \left[\frac{1}{C} - \left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right)\right] L\cos\vartheta + R\,[\sin\nu \cot\vartheta \cos(\varphi-\mu) - \cos\nu] \\ \vartheta^{\cdot} &= \left(\frac{1}{A} - \frac{1}{B}\right) L \sin\vartheta \sin\varphi\cos\varphi + R\sin\nu\sin(\varphi-\mu) \\ \psi^{\cdot} &= \left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right) L - R\sin\nu\,\frac{\cos(\varphi-\mu)}{\sin\vartheta}\end{aligned} \tag{1.13}$$

and its integral (1.5), as

$$\begin{aligned}&\left[\frac{1}{C} - \left(\frac{\sin^2\varphi}{A} + \frac{\cos^2\varphi}{B}\right)\right] L\sin^2\vartheta + \\ &+ 2R\,[\sin\nu\sin\vartheta\cos(\varphi-\mu) + \cos\nu\cos\vartheta] = \text{const}\end{aligned} \tag{1.14}$$

Next, limiting ourselves to the case when the carrier has axial dynamic symmetry, i.e. $A = B$, we can write (1.13) and its integral (1.14), as

$$\varphi^{\cdot} = \left(\frac{1}{C} - \frac{1}{A}\right) L\cos\vartheta + R\,[\sin\nu \cot\vartheta\cos(\varphi-\mu) - \cos\nu] \tag{1.15}$$

$$\vartheta^{\cdot} = R\sin\nu\sin(\varphi-\mu), \qquad \psi^{\cdot} = \frac{L}{A} - R\sin\nu\,\frac{\cos(\varphi-\mu)}{\sin\vartheta}$$

$$a\sin^2\vartheta + \sin\nu\sin\vartheta\cos(\varphi-\mu) + \cos\nu\cos\vartheta = c$$

$$(a = L/2R\ (1/C - 1/A)) \tag{1.16}$$

Here c is the constant of integration. Using (1.16) we can obtain

$$\vartheta^{\cdot} = \frac{R}{\sin\vartheta}\sqrt{\sin^2\nu\sin^2\vartheta - (c - a\sin^2\vartheta - \cos\nu\cos\vartheta)^2} \tag{1.17}$$

which can be represented as

$$\left(\frac{d\cos\vartheta}{dt}\right)^2 = -R^2 f_1(\vartheta) f_2(\vartheta) \tag{1.18}$$

Here, functions $f_1(\vartheta)$ and $f_2(\vartheta)$, which play a major part in the following investigation are, obviously

$$f_1(\vartheta) = a\sin^2\vartheta + \cos(\vartheta-\nu) - c, \qquad f_2(\vartheta) = a\sin^2\vartheta + \cos(\vartheta+\nu) - c \tag{1.19}$$

We see at once from (1.18), that motion of the body in the ϑ-direction is possible only within those ranges of values of the angle ϑ, in which the functions f_1 and f_2 have opposite signs. Circles on the unit sphere on which either $f_1 = 0$ or $f_2 = 0$, represent the boundaries of these ranges.

2. Investigation of the $f_1(\vartheta)$ and $f_2(\vartheta)$ functions. From (1.19) we see, that between f_1 and f_2 following relationships exist (2.1)

$$f_1(\vartheta) = f_2(-\vartheta), \qquad f_1(\vartheta, \nu) = f_2(\vartheta, -\nu), \qquad f_1(\vartheta, a, c) = -f_2(\pi - \vartheta, -a, -c)$$

We also see that the end-points of the interval $0 \leqslant \vartheta \leqslant \pi$, f_1 and f_2 have the same values, while their derivatives with respect to ϑ, are of the same magnitude, but opposite sign. The relationship (2.1) shows, that it is sufficient to investigate only one of these functions over the interval $-\pi \leqslant \vartheta \leqslant \pi$. Using (1.16), we can represent f_1 and f_2 as (2.2)

$$f_1 = \sin\nu \sin\vartheta\,[1 - \cos(\varphi - \mu)], \qquad f_2 = -\sin\nu\sin\vartheta\,[1 + \cos(\varphi - \mu)]$$

from which we see, that on the boundaries of the regions of possible motion where f_1 or f_2 becomes equal to zero, $\varphi = \mu \pm k\pi$ (where k is an integer). Apart from that, (2.2) implies that if the same function (either f_1 or f_2) becomes equal to zero on both circles bounding the region of possible motions of the axis of the body, then the motion will be oscillatory with respect to the angle φ, while, if on one of the circles $f_1 = 0$ and on the other $f_2 = 0$, then the motion will be gyratory in φ. During the motion of the z-axis from one boundary circle to the other, angle φ will change by π. From the relation $f_1 - f_2 = 2\sin\nu\sin\vartheta$ it follows that the curves $f_1(\vartheta)$ and $f_2(\vartheta)$ intersect only at the end-points of the interval $0 \leqslant \vartheta \leqslant \pi$, forming a closed, snail-like figure (Fig.1). Motion in ϑ is possible only on this segment of the ϑ-axis, which is inside this figure. Its position relative to the ϑ-axis is determined by the magnitude of the constant c, i.e. by the initial conditions of the motion, while its shape and size depend on the parameters a and ν. When $\nu = 0$ (which corresponds to the rotation of the flywheel only about the z-axis of symmetry of the body), the curves $f_1(\vartheta)$ and $f_2(\vartheta)$ coincide, i.e. the plane figure degenerates into a single line. In this case, the carrier will execute a regular precession, and

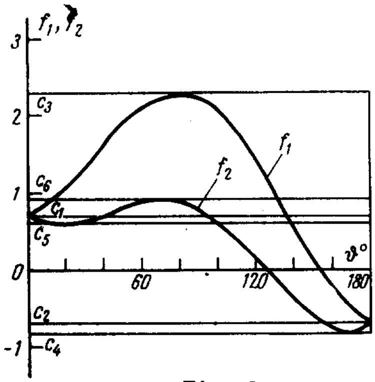


Fig. 1

$$\cos\vartheta = \frac{1}{2a} \pm \left(\frac{1}{4a^2} - \frac{c-a}{a}\right)^{1/2}$$

When $\nu = \frac{1}{2}\pi$ (i.e. the projection of the kinetic moment of the flywheels on the z-axis is equal to zero), the curves $f_1(\vartheta)$ and $f_2(\vartheta)$, and consequently the figure and the whole pattern of motion of the z-axis of the body, become symmetric with respect to $\vartheta = \pi/2$. In general, as seen from Fig.1, depending on the number and distribution of extremas of the functions $f_1(\vartheta)$ and $f_2(\vartheta)$, the region of possible motions of the z-axis in ϑ may consist either of one strip $\vartheta_1 \leqslant \vartheta \leqslant \vartheta_2$, or of two strips $\vartheta_1 \leqslant \vartheta \leqslant \vartheta_2$ and $\vartheta_3 \leqslant \vartheta \leqslant \vartheta_4$. To show the influence of the parameters a and ν on the shape and position of the figure, we shall investigate the function $f_2(\vartheta)$. Its extrema are given by

$$a \sin 2\vartheta = \sin(\vartheta + \nu) \tag{2.3}$$

Investigation of solutions of this equation for various a and ν is best performed by analysis of graphs representing its left- and right-hand sides by separate sine curves differing from each other in amplitude, phase and frequency. Using such a graph it is easy to establish that, depending on the values of a and ν, (2.3) can have either one or three roots. In the limiting case when the sine curves are tangent to each other, (2.3) has two roots. To find the position of this limiting curve on the parametric $a\nu$-plane, we shall denote the values of a, ν and ϑ, corresponding to the condition of tangency of the sine curves, by a_*, ν_* and ϑ_*. Then, assuming that at the tangent point the derivatives of both sine curves are the same, we have

$$a_* \sin 2\vartheta_* = \sin(\vartheta_* + \nu_*), \qquad 2a_* \cos 2\vartheta_* = \cos(\vartheta_* + \nu_*) \tag{2.4}$$

Eliminating ϑ_*, we can obtain the explicit equation of the boundary line, although in case of numerical computations, parametric representation of this line

$$\tan \nu_* = \tan^3 \vartheta_*, \qquad a_* = \frac{\sin(\vartheta_* + \nu_*)}{\sin 2\vartheta_*} \tag{2,5}$$

is more convenient.

Fig.2 shows this line Γ, dividing the parametric $a\nu$-plane into the regions corresponding to one root of (2.3) (non-shaded region) and to three roots (shaded regions). Coordinates of the series of points of this line are

$\nu_* = 0°$	0°17′	2°45′	10°53′	30°35′	45°	59°24′	79°04′	87°12′	90°
$\lvert a_*\rvert = 0.5$	0.5219	0.6016	0.7578	0.9577	1.0	0.9578	0.7576	0.6028	0.5

Two roots of Equation (2.3) correspond to the points of transition from one region to the other. By the second condition of (2.1) we can assume that the right-hand side semi-plane of the $a\nu$-plane characterizes the function $f_1(\vartheta)$, while the left-hand side, the function $f_2(\vartheta)$. Therefore, the extrema of these functions are given by the points distributed in the left- and right-hand side semi-plane symmetrically with respect to the a-axis. Fig.2 also shows that the function $f_1(\vartheta)$ can have three extrema only, when $a < -0.5$, while for $f_2(\vartheta)$ the corresponding condition is, that $a > 0.5$. Assuming further that $\nu \geq 0$ (otherwise we interchange f_1 and f_2), we have $df_1/d\vartheta \geqslant 0$ when $\vartheta = 0$, i.e. the extrema of $f_1(\vartheta)$ occur in the following order: maximum, minimum and maximum, while $f_2(\vartheta)$ will have minimum, maximum and minimum (provided all three exist).

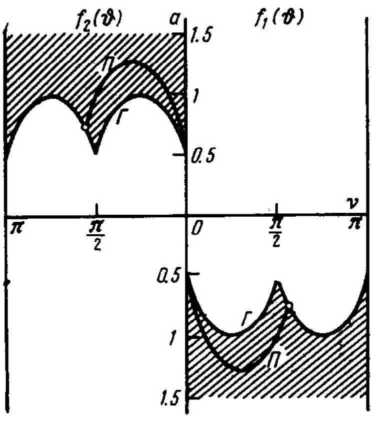


Fig. 2

The question whether $\max f_2 > f_2(0)$ is of the major importance to the motion of the body in φ. Obviously, $\max f_2 = f_2(0)$ is the boundary case. Denoting the values of a, ν and ϑ, corresponding to this case a_{**}, ν_{**} and ϑ_{**}, we can write the corresponding relationship in the form

$$a_{**} \sin 2\vartheta_{**} - \sin(\vartheta_{**} + \nu_{**}) = 0$$
$$a_{**} \sin^2 \vartheta_{**} + \cos(\vartheta_{**} + \nu_{**}) - \cos \nu_{**} = 0 \tag{2.6}$$

Eliminating ϑ_{**}, we obtain the equation of the line Π dividing the $a\nu$-plane into the regions, in which $\max f_2 > f_2(0)$ and $\max f_2 < f_2(0)$. Obviously, this line should lie completely within the shaded region, since when only one extremum of $f_2(\vartheta)$ exists, it has no maximum at all. Parametric representation of this line

$$\tan \nu_{**} = \frac{2(1 - \cos \vartheta_{**})^2}{\sin 2\vartheta_{**}}, \qquad a_{**} = \frac{\sin(\vartheta_{**} + \nu_{**})}{\sin 2\vartheta_{**}} \tag{2.7}$$

is convenient for its construction.

Coordinates of points on this line are

$\nu_{**} = 0°$	0°37′	2°20′	6°21′	14°33′	30°01′	45°	53°23′	75°55′	84°03′	90°	100°53′
$\lvert a_{**}\rvert = 0.5$	0.548	0.618	0.735	0.917	1.155	1.276	1.299	1.193	1.099	1.0	0.7558

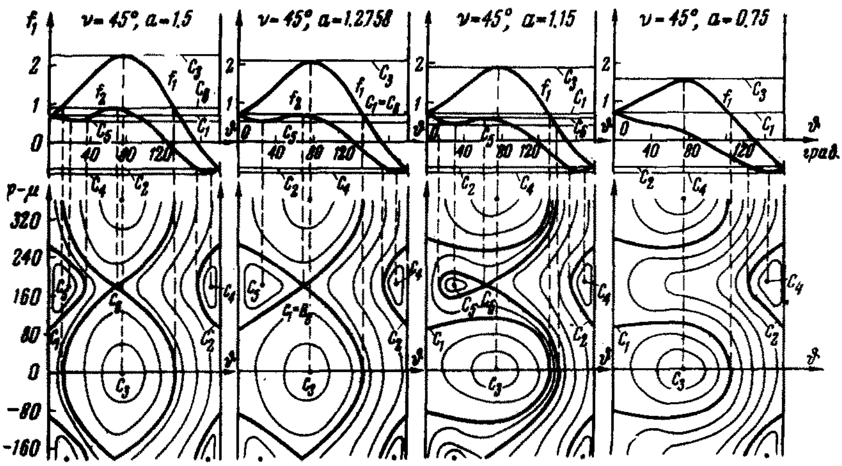
From Fig.2 we see, that lines Γ and Π have two common points. First of them is, obviously, $|a| = 0.5$, $\nu = 0$. The other is easily found by comparing (2.5) and (2.7) which, together yield the following expression for ϑ

$$\tan^3 \vartheta = \frac{2(1 - \cos \vartheta)^2}{\sin 2\vartheta} \tag{2.8}$$

Hence $\vartheta = 120°, \tan\nu = -3\sqrt{3}$; $\nu = 100°53'$; $|a| = 0.7558$. For these values, point of inflection and the point $f_2(0)$ are at the same level.

Consequently, it follows from Fig.2 that the line π separates, out of the region of existence of three extrema of $f_2(\vartheta)$ a segment, corresponding to the condition $\max f_2 < f_2(0)$. The behavior of $f_1(\vartheta)$ is clearly analogous to that of $f_2(\vartheta)$ and corresponds only to the condition $a < 0$.

3. Investigation of phase trajectories. The investigation of phase trajectories in the $\vartheta\varphi$-plane can be performed with relatively little difficulty owing to the presence of the integral (1.16). It should however be borne in mind, that the general pattern of behavior of phase trajectories corresponding to various values of c, depends essentially on the region of the parametric plane, in which the values of a and ν lie. Fig.3 shows the



a b c d

Fig. 3

change of pattern of the phase plane when the characteristic point on the $a\nu$-plane is transferred, first, from the region above the line π into the region situated between π and Γ, and then into the region situated under the Γ line (along the straight line $\nu = 45°$). Fig.3a constructed for the values $\nu = 45°$ and $a = 1.5$ shows, that the parameter c of the family of phase trajectories can, in general, assume six critical values. First two which are $c_1 = \cos\nu$ and $c_2 = -\cos\nu$ correspond to the case when angle ϑ can assume the values 0 or π. Trajectories corresponding to these values of c, represent the dividing lines separating the regions of existence of the types of motion oscillatory in φ, from those of the gyratory motion. In the limit, the oscillatory cycles enveloped by the dividing lines degenerate into equilibrium points with the values $c = c_4$ and $c = c_5$. Finally, the remaining cycle has its center defined by the critical value $c = c_3$, and its outer boundary is given by the dividing line $c = c_6$, consisting of two branches. Equations of the dividing lines $c = c_1$ and $c = c_2$ are

$$\cos\varphi = \frac{1}{\sin\nu}\tan\frac{\vartheta}{2}\left(\cos\nu - 2a\cos^2\frac{\vartheta}{2}\right)$$
$$\cos\varphi = -\frac{1}{\sin\nu}\cot\frac{\vartheta}{2}\left(\cos\nu + 2a\sin^2\frac{\vartheta}{2}\right) \tag{3.1}$$

Fig.3b corresponds to the case, when the characteristic point belonging to the parameteric plane, lies on the boundary line Π . Here the dividing lines $c = c_6$ and $c = c_1$ coincide, therefore the gyratory modes of motion in φ are found possible only in the region between the lines $c = c_2$ and $c = c_6$. In this case, equation of the dividing line $c = c_1$ coincident with $c = c_6$ can be written as

$$\cos\varphi = \tan\frac{\vartheta}{2}\cot\frac{\vartheta_{**}}{2}\left(\frac{\cos\vartheta_{**} - \cos\vartheta}{\sin^2\vartheta_{**}} - 1\right) \tag{3.2}$$

If the characteristic point is further displaced into the region contained between the Π and Γ curves (Fig.3c), then the dividing line $c = c_1$ no longer encloses the center $c = c_5$, but $c = c_3$ instead. Center $c = c_5$, is, on the other hand, enclosed by the loop of the dividing line $c = c_6$. As the characteristic point lying on the $a\nu$-plane approaches the line Γ , the loop around the center $c = c_5$ becomes smaller, and disappears completely at the moment, at which the point crosses Γ . Phase pattern then assumes the form shown on the Fig. 3d.

We can see from the above figures, that the snail-like figure represents sufficient means of obtaining the qualitative picture of types of motion of the body with respect to the angle ϑ and φ . Of course, necessary values of parameters a and ν must be chosen on the $a\nu$-plane (Fig.2). In the number of cases however, determination of the law of motion of the z-axis of the body in space, is also necessary. This is equivalent to finding a trajectory described by this axis on the surface of a fixed sphere, the center of which coincides with the center of inertia of the body. To do this, we shall write the equation for ψ , as

$$\psi^{\cdot} = \frac{L}{A} + \frac{R}{2}\frac{f_1(\vartheta) + f_2(\vartheta)}{\sin\vartheta} \tag{3.3}$$

and shall express the tangent of the angle χ formed by a tangent to the trajectory of the z-axis and the local parallel to the unit sphere

$$\tan\chi = \frac{d\vartheta}{\sin\vartheta\, d\psi} = \frac{\sqrt{-f_1(\vartheta) f_2(\vartheta)}}{(L/AR)\sin^2\vartheta + {}^1/_2\,[f_1(\vartheta) + f_2(\vartheta)]} \tag{3.4}$$

From this it follows that $\tan\chi$ becomes equal to zero on the limiting circles enclosing regions of possible motions, and becomes infinite, when $u (u = \cos \vartheta)$,

$$u = \frac{R\cos\nu \pm \sqrt{[Rc - L(1/C + 1/A)]^2 + R^2(\cos^2\nu - c^2)}}{L(1/C + 1/A)} \tag{3.5}$$

When $c = \pm\cos\nu$, we obviously have $u \pm 1$, i.e. z-axis of the body passes through the pole of the unit sphere. We should also note that, if one of the roots of the denominator of the fraction (3.4) coincides with one of the roots of its numerator, then the trajectory of the z-axis has, on the corresponding boundary circle, a cuspal point. Having the expression for $\tan\chi$, we can easily obtain the expression for the "curvature" of the trajectory

$$K = \frac{d\chi}{\sqrt{1 + (\sin\vartheta\, d\psi / d\vartheta)^2}\, d\vartheta} = \frac{1}{2} \frac{1}{(1 + \tan^2\chi)^{3/2}} \frac{d \tan\chi}{d\vartheta} \qquad (3.6)$$

Returning to Equation (3.3) we see that, when $R = 0$ or $\nu = 0$, the body exhibits regular precession. If, on the other hand, $L = 0$, then the body exhibits pure rotation about a fixed axis, and its z-axis describes, on the unit sphere, a circle of solid angle ν. This agrees with the results given in [4].

In general, when all the parameters given above are different from zero, four types of motion are possible. We shall consider them, using the functions $f_1(\vartheta)$ and $f_2(\vartheta)$, given in Fig.1, where $L / AR = 0.6$, $C / A = 1/_6$.

First type of motion corresponds to $\sigma_6 < \sigma < \sigma_3$ when function $f_1(\vartheta)$ becomes equal to zero on both, upper and lower circles. The values of $\psi^\cdot$ on the limit circles, $\psi_+^\cdot$ on the upper and $\psi_-^\cdot$ on the lower circle will be

$$\psi_+^\cdot = \frac{L}{A} - \frac{R \sin\nu}{\sin\vartheta_1}, \qquad \psi_-^\cdot = \frac{L}{A} - \frac{R\sin\nu}{\sin\vartheta_2} \qquad (3.7)$$

Assuming that L and R are always positive we see, that the velocity of motion on each of the boundary circles can be either positive, or negative. The trajectory contained in the strip $\vartheta_1 \leqslant \vartheta \leqslant \vartheta_2$ may have loops and cuspal points (Figs.4a and b).

Second and third type of motion takes place when $\sigma_1 < \sigma < \sigma_6$, if $\vartheta_1 \leqslant \vartheta_0 \leqslant \vartheta_2$ and when $\sigma_2 < \sigma < \sigma_6$ if $\vartheta_3 \leqslant \vartheta_0 \leqslant \vartheta_4$. The corresponding expressions for $\psi^\cdot$ are

$$\psi_+^\cdot = \frac{L}{A} - \frac{R\sin\nu}{\sin\vartheta_1}, \qquad \psi_-^\cdot = \frac{L}{A} + \frac{R\sin\nu}{\sin\vartheta_2} \qquad (3.8)$$

$$\psi_+^\cdot = \frac{L}{A} + \frac{R\sin\nu}{\sin\vartheta_3}, \qquad \psi_-^\cdot = \frac{L}{A} - \frac{R\sin\nu}{\sin\vartheta_4} \qquad (3.9)$$

It is easy to see that the loops are formed only on the circles facing the poles of the unit sphere (Fig.4c).

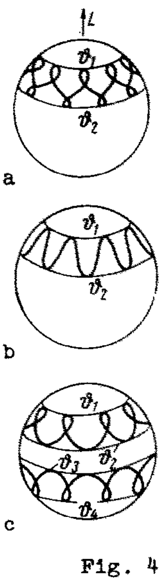
The fourth type will be characterized by the values $\sigma_5 < \sigma < \sigma_1$ or $\sigma_4 < \sigma < \sigma_2$, so that for $\psi^\cdot$ we have

$$\psi_+^\cdot = \frac{L}{A} + \frac{R\sin\nu}{\sin\vartheta_1}, \qquad \psi_-^\cdot = \frac{L}{A} + \frac{R\sin\nu}{\sin\vartheta_2} \qquad (3.10)$$

where we note the absence of loops on either boundary circle. Finally, in the limiting case when $\sigma = \pm\cos\nu$, the equations for $\psi^\cdot$ and $\vartheta^\cdot$ become

$$\psi^\cdot = \left(\frac{1}{C} + \frac{1}{A}\right)\frac{L}{2} - \frac{R\cos\nu}{1 + \cos\vartheta}$$

$$\vartheta^\cdot = R\sqrt{\sin^2\nu - (\cos\nu \tan{}^1/_2\,\vartheta - a\sin\vartheta)^2} \qquad (3.11)$$



a b c

Fig. 4

We see that the velocity $\vartheta^\cdot$ does not vanish when $\vartheta = 0$ and axis of the body passes through the pole of the unit sphere without stopping, while the angle ψ undergoes a discontinuous change of $\pm\pi$.

The above considerations refer to the motion of the axis of a spindle-shaped body, for which $a > 0$. However, using last of the relations of (2.1) it is easy to show, that for a disk-shaped body ($a < 0$), the types of motion obtained will be identical.

4. Integration of equations of motion. In order to integrate the system (1.15), let us write the quadrature for the angle ϑ , as

$$\int \frac{du}{\sqrt{a^2 (u_1 - u)(u - u_2)(u - u_3)(u - u_4)}} = \int R\, dt \tag{4.1}$$

The polynomial under the radical sign is

$$P(u) = \sin^2 \nu\,(1 - u^2) - [a\,(1 - u^2) + u \cos \nu - c]^2 \tag{4.2}$$

By considering the values of this polynomial at the points $u = \pm 1$ and $u = 0$, we can obtain the relations connecting the roots of the polynomial and the parameters a, ν and c . They are

$$a^2 p_1 = (\cos \nu - c)^2, \qquad a^2 p_2 = (\cos \nu + c)^2, \qquad a^2 p_3 = (a - c)^2 - \sin^2 \nu \tag{4.3}$$

where

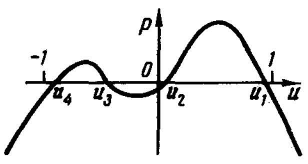


Fig. 5

$$p_1 = (1 - u_1)(1 - u_2)(1 - u_3)(1 - u_4)$$
$$p_2 = (1 + u_1)(1 + u_2)(1 + u_3)(1 + u_4)$$
$$p_3 = u_1 u_2 u_3 u_4 \quad (p_1 \geqslant 0,\ p_2 \geqslant 0) \tag{4.4}$$

From (4.3) we find (4.5)

$$a = \pm \frac{1 - \lambda}{Q}, \quad \cos \nu = \pm \frac{\lambda \sqrt{p_1}}{Q}, \quad c = \pm \frac{\sqrt{p_1}}{Q}$$

where

$$Q = \sqrt{(1 - p_3)(1 - \lambda)^2 + p_1 (1 + \lambda^2) \pm 2 \sqrt{p_1}\,(1 - \lambda)}$$

$$\lambda_1 = \frac{\sqrt{p_2} + \sqrt{p_1}}{\sqrt{p_2} - \sqrt{p_1}}, \qquad \lambda_2 = \frac{\sqrt{p_2} - \sqrt{p_1}}{\sqrt{p_2} + \sqrt{p_1}} \tag{4.6}$$

For λ we choose the value λ_1 or λ_2 , for which $|\cos \nu| \leq 1$. The relations obtained allow us to choose the parameters necessary for the realization of a given type of the carrier.

Let us integrate Equation (4.1). We should note that the polynomial $P(u)$ is always negative when $|u| > 1$, hence all its real roots are contained in the interval $-1 \leq u \leq 1$ (Fig.5). Assuming for example that only two roots of $P(u)$ are real and, that the motion takes place within the strip $u_2 \leq u \leq u_1$ we can, using the notation of [5], write the integral of (4.1) as

$$\int_{u_2}^{u} \frac{du}{\sqrt{P(u)}} = \frac{1}{\sqrt{pq}} F(\phi, k) \tag{4.7}$$

$$P(u) = a^2 [(u - m)^2 + n^2](u_1 - u)(u - u_2), \qquad \phi = 2 \tan^{-1} \left[\frac{q\,(u_1 - u)}{p\,(u - u_2)}\right]^{1/2}$$

$$p^2 = (m - u_1)^2 + n^2, \qquad q^2 = (m - u_2)^2 + n^2, \qquad k = \frac{1}{2} \left[\frac{(u_1 - u_2)^2 - (p - q)^2}{pq}\right]^{1/2}$$

Here $F(\phi, k)$ is the elliptic integral of the first kind. Assuming, that the process of motion starts at some initial value u_0 which, in general, is not a root of $P(u)$, we have

$$\int_{u_0}^{u} \frac{du}{\sqrt{P(u)}} = \frac{1}{\sqrt{pq}} [F(\phi_0, k) - F(\phi_0, k)] \tag{4.8}$$

from which we have

$$\sin \phi = sn\,(\tau \sqrt{pq} + F_0), \qquad \cos \phi_0 = cn\,(\tau \sqrt{pq} + F_0) \tag{4.9}$$

and finally we obtain

$$u(\tau)=\frac{pu_2+qu_1+(pu_2-qu_1).cn\,(\tau\sqrt{pq}+F_0)}{p+q+(p-q)\,cn\,(\tau\sqrt{pq}+F_0)}\qquad (F_0=F(\phi_0,k))\tag{4.10}$$

Solution for the case of four real roots of $P(u)$ can be constructed in an analogous manner. To find the explicit expression for the angle ψ, we must substitute the expression obtained for $u(\tau)$ into (3.3), written in the form

$$2\psi^{\cdot}=\left(\frac{1}{A}+\frac{1}{C}\right)L+R\left(\frac{\cos\nu-c}{1-u}-\frac{\cos\nu+c}{1+u}\right)\tag{4.11}$$

Then, angle ψ can be represented as a sum of a linear function τ and of integrals of the form

$$\int\frac{d\tau}{cn\,(\tau+F_0)+b},\qquad \int\frac{d\tau}{sn\,(\tau+F_0)+b}\tag{4.12}$$

which can be expressed in terms of theta-function.

In conclusion we shall note that for an asymmetric body, e.g. when $A\neq B\neq C$, solution can be represented in an analogous form, if we assume that $\mu=0$.

Indeed, in this case the integral (1.14) will yield

$$\cos\varphi=\frac{-R\sin\nu\pm\sqrt{\alpha'u^2+2\beta'u+\gamma'}}{(1/A-1/B)\,L\sin\vartheta}\tag{4.13}$$

Using this to eliminate the angle φ from the second equation of (1.13), we arrive at the following integral for ϑ:

$$\int\frac{du}{\sqrt{\alpha'u^2+2\beta'u+\gamma'}\sqrt{\alpha''u^2+2\beta''u+\gamma''+2\delta''\sqrt{\alpha'u^2+2\beta'u+\gamma'}}}=\int R\,dt\tag{4.14}$$

$$\begin{gathered}\alpha'=L^2\left(\frac{1}{A}-\frac{1}{B}\right)\left(\frac{1}{C}-\frac{1}{A}\right),\quad \beta'=-LR\cos\nu\left(\frac{1}{A}-\frac{1}{B}\right)\\ \gamma'=R^2\sin^2\nu+cL\left(\frac{1}{A}-\frac{1}{B}\right)-L^2\left(\frac{1}{A}-\frac{1}{B}\right)\left(\frac{1}{C}-\frac{1}{A}\right)\\ \alpha''=L^2\left(\frac{1}{B}-\frac{1}{A}\right)\left(\frac{1}{C}-\frac{1}{B}\right),\quad \beta''=LR\cos\nu\left(\frac{1}{A}-\frac{1}{B}\right)\\ \gamma''=-2R^2\sin^2\nu-cL\left(\frac{1}{A}-\frac{1}{B}\right)+L^2\left(\frac{1}{A}-\frac{1}{B}\right)\left(\frac{1}{C}-\frac{1}{B}\right),\quad \delta''=R\sin\nu\end{gathered}\tag{4.15}$$

Here c is the constant of integration from (1.14). The integral (4.14) can, by change of variables, be reduced to an elliptic integral. Assuming for example that $\alpha'>0$, we shall use the Euler substitution

$$\sqrt{\alpha'u^2+2\beta'u+\gamma'}=\sqrt{\alpha'}\,u+v\tag{4.16}$$

from which, squaring and differentiating we obtain

$$\frac{du}{u\sqrt{\alpha'}+v}=\frac{dv}{\beta'-v\sqrt{\alpha'}}\tag{4.17}$$

as a result of which, the integral (4.14) assumes the form (4.18)

$$\int\frac{dv}{\sqrt{{}^1/_4\alpha''(v^2-\gamma')^2+(v^2-\gamma')(\beta'-v\sqrt{\alpha'})(\beta''+\delta''\sqrt{\alpha'})+(\gamma''+v\delta'')(\beta'-v\sqrt{\alpha'})^2}}$$

Here, under the radical sign, we have a 4-th degree polynomial, hence all the transformations that follow, will be analogous to the previous ones. To find the angle ψ, it remains to replace, in (4.11), u by v. This will result in an integral of the type of (4.12).

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